Pensieve header: The free-Lie meta-crossed-product structure, lazy evaluation version.

```mathematica
SetDirectory["C:\\drorbn\\AcademicPensieve\\2012-07"];
<< "../Projects/FreeLie/FreeLieLE.m"

Randomλ[d_, m_, n_] := Module[{bas},
   bas = AllLyndonWords[{d}, Characters[StringTake["1234567890", m]]];
   λ[
    MakeCWSeries[RandomInteger[{-2, 2}, Length[bas]].bas /. LW → CW],
    Sum[h[j] MakeLieSeries[RandomInteger[{-2, 2}, Length[bas]].bas], {j, n}]
   ]
  ];
hL[λ_] := Union[Cases[λ, h[s_] :> s, Infinity]];
λ[ω_, μ_][d_] := λ[ω[d], μ /. s_LieSeries :> s[d]];
λ[ω1_, μ1_] ≡ λ[ω2_, μ2_] :=
  ω1 ≡ ω2 && (And @@ ((D[μ1, h[#]] ≡ D[μ2, h[#]]) & /@ hL[{μ1, μ2}]));
LieDerivation[der_][λ[ω_, μ_]] := λ[ω // der, Collect[μ, _h, der]];
LieMorphism[mor_][λ[ω_, μ_]] := λ[ω // mor, Collect[μ, _h, mor]];

{Randomλ[1, 3, 3], Randomλ[3, 3, 3]}
```

$\{\lambda[\text{CWS}[\text{CW}[1] - \text{CW}[2] + \text{CW}[3], 0, 0], h[1] \text{LS}[-2\langle1\rangle - 2\langle3\rangle, 0, 0] +$
$\quad h[2] \text{LS}[-2\langle1\rangle + 2\langle2\rangle + 2\langle3\rangle, 0, 0] + h[3] \text{LS}[-2\langle1\rangle - \langle2\rangle + 2\langle3\rangle, 0, 0]],$
$\lambda[\text{CWS}[\text{CW}[1] - \text{CW}[2] + 2\text{CW}[3], -\text{CW}[13] - \text{CW}[23],$
$\quad 2\text{CW}[112] - 2\text{CW}[113] + 2\text{CW}[122] - \text{CW}[123] - 2\text{CW}[132] + \text{CW}[133] + \text{CW}[223]],$
$\quad h[1] \text{LS}[-2\langle1\rangle + \langle2\rangle, 2\langle12\rangle + 2\langle13\rangle - 2\langle23\rangle, 2\langle112\rangle - \langle113\rangle + \langle122\rangle - \langle123\rangle +$
$\qquad 2\langle132\rangle + \langle133\rangle + \langle223\rangle] + h[2] \text{LS}[-\langle1\rangle + \langle3\rangle, -\langle12\rangle + \langle13\rangle + \langle23\rangle,$
$\qquad \langle112\rangle - 2\langle113\rangle + \langle122\rangle - 2\langle123\rangle - 2\langle132\rangle + 2\langle133\rangle + \langle233\rangle] + h[3]$
$\quad \text{LS}[-\langle1\rangle - 2\langle2\rangle + \langle3\rangle, 2\langle12\rangle + 2\langle23\rangle, -2\langle112\rangle + \langle113\rangle + \langle122\rangle + \langle133\rangle - \langle223\rangle - \langle233\rangle]]\}$

```mathematica
tm[x_, y_, z_][λ[ω_, μ_]] := λ[ω, μ] // LieMorphism[{⟨x⟩ → ⟨z⟩, ⟨y⟩ → ⟨z⟩}];
hm[x_, y_, z_][λ[ω_, μ_]] := λ[ω, Plus[
    μ /. {h[x] → 0, h[y] → 0},
    h[z] BCH[D[μ, h[x]], D[μ, h[y]]]
   ]];
hta[x_, y_, z_][λ[ω_, μ_]] := λ[0,
   Collect[μ, _h,
    StableApply[
      LieMorphism[{LW[y] → Ad[ScaleLieSeries[-1, D[μ, h[x]]]][LW[z]]}], #] &
   ]
  ];
hta[x_, y_][λ[ω_, μ_]] := Module[{mor, ω0, μ0},
   {ω0, μ0} = List @@ hta[x, y, LW["z"]][λ[ω, μ]];
   mor = LieMorphism[{LW["z"] → LW[y]}];
   λ[ω0, Collect[μ0, _h, mor]]
  ];
dm[x_, y_, z_][λ_] := λ // hta[y, x] // tm[x, y, z] // hm[x, y, z];
Rp[x_, y_] := λ[0, h[y] MakeLieSeries[⟨x⟩]];
```

Testing tm

```
n = $SeriesShowDegree = $SeriesCompareDegree = 3;
Print /@ {λ0 = Randomλ[n, 4, 1],
   λ0 // tm[1, 2, 2],
   λ0 // tm[2, 3, 3],
   t1 = λ0 // tm[1, 2, 2] // tm[2, 3, 3],
   t2 = λ0 // tm[2, 3, 3] // tm[1, 3, 3],
   t1 ≡ t2
   };
```

λ[CWS[CW[2] + 2 CW[3] + CW[4], 2 CW[12] + 2 CW[14] − 2 CW[23] − CW[24] + 2 CW[34],
  − CW[112] − CW[122] + 2 CW[123] + 2 CW[133] + 2 CW[134] + 2 CW[142] + CW[143] −
  2 CW[144] − CW[223] + CW[224] − 2 CW[233] − 2 CW[243] − 2 CW[244] − 2 CW[344]],
 h[1] LS[−2 ⟨2⟩ − ⟨3⟩ − ⟨4⟩, −2 ⟨12⟩ + 2 ⟨13⟩ − 2 ⟨14⟩ − 2 ⟨23⟩ + ⟨24⟩ + ⟨34⟩,
  2 ⟨112⟩ − ⟨113⟩ − ⟨114⟩ − 2 ⟨122⟩ − ⟨123⟩ − 2 ⟨124⟩ + 2 ⟨132⟩ − 2 ⟨133⟩ − 2 ⟨134⟩ + 2 ⟨142⟩ +
   ⟨143⟩ + ⟨144⟩ + 2 ⟨223⟩ + 2 ⟨224⟩ + ⟨233⟩ − 2 ⟨234⟩ + 2 ⟨243⟩ − 2 ⟨244⟩ − 2 ⟨334⟩ − ⟨344⟩]]

λ[CWS[CW[2] + 2 CW[3] + CW[4], 2 CW[22] − 2 CW[23] + CW[24] + 2 CW[34],
  − 2 CW[222] + CW[223] + 3 CW[224] + 2 CW[234] − CW[243] − 4 CW[244] − 2 CW[344]],
 h[1] LS[−2 ⟨2⟩ − ⟨3⟩ − ⟨4⟩, −⟨24⟩ + ⟨34⟩,
  −2 ⟨223⟩ − 3 ⟨224⟩ − ⟨233⟩ − 4 ⟨234⟩ + 3 ⟨243⟩ − ⟨244⟩ − 2 ⟨334⟩ − ⟨344⟩]]

λ[CWS[3 CW[3] + CW[4], 2 CW[13] + 2 CW[14] − 2 CW[33] + CW[34],
  − CW[113] + 3 CW[133] + 2 CW[134] + 3 CW[143] − 2 CW[144] − 3 CW[333] − CW[334] − 4 CW[344]],
 h[1] LS[−3 ⟨3⟩ − ⟨4⟩, −2 ⟨14⟩ + 2 ⟨34⟩,
  ⟨113⟩ − ⟨114⟩ − 2 ⟨133⟩ − 4 ⟨134⟩ + 3 ⟨143⟩ + ⟨144⟩ − 4 ⟨334⟩ − 3 ⟨344⟩]]

λ[CWS[3 CW[3] + CW[4], 3 CW[34], − CW[333] + 4 CW[334] − 6 CW[344]],
 h[1] LS[−3 ⟨3⟩ − ⟨4⟩, 0, −12 ⟨334⟩ − 2 ⟨344⟩]]

λ[CWS[3 CW[3] + CW[4], 3 CW[34], − CW[333] + 4 CW[334] − 6 CW[344]],
 h[1] LS[−3 ⟨3⟩ − ⟨4⟩, 0, −12 ⟨334⟩ − 2 ⟨344⟩]]

True

```
t1 = λ0 // tm[1, 2, 2] // tm[2, 3, 3] // InputForm
```

λ[LieSeries[LieMorphismOnLieSeries$110], h[1]*LieSeries[LieMorphismOnLieSeries$112]]

Testing hm

```
Print /@ {λ0 = Randomλ[4, 2, 3],
   λ0 // hm[1, 2, 2],
   t1 = λ0 // hm[1, 2, 2] // hm[2, 3, 3],
   t2 = λ0 // hm[2, 3, 3] // hm[1, 3, 3],
   t1 ≡ t2
   };
```

λ[CWS[CW[1] + CW[2], CW[12], CW[112] + CW[122]], h[1] LS[2 ⟨1⟩ + ⟨2⟩, ⟨12⟩, ⟨112⟩ + 2 ⟨122⟩] +
   h[2] LS[-2 ⟨1⟩ + 2 ⟨2⟩, ⟨12⟩, 2 ⟨112⟩] + h[3] LS[-⟨1⟩ - ⟨2⟩, 0, 0]]

λ[CWS[CW[1] + CW[2], CW[12], CW[112] + CW[122]],
 h[2] LS[3 ⟨2⟩, 5 ⟨12⟩, 7 ⟨112⟩ + 3 ⟨122⟩] + h[3] LS[-⟨1⟩ - ⟨2⟩, 0, 0]]

$$\lambda\left[\text{CWS}[\text{CW}[1] + \text{CW}[2], \text{CW}[12], \text{CW}[112] + \text{CW}[122]],\right.$$
$$\left. h[3] \text{LS}\left[-⟨1⟩ + 2 ⟨2⟩, \frac{13 ⟨12⟩}{2}, \frac{39 ⟨112⟩}{4} - \frac{⟨122⟩}{2}\right]\right]$$

$$\lambda\left[\text{CWS}[\text{CW}[1] + \text{CW}[2], \text{CW}[12], \text{CW}[112] + \text{CW}[122]],\right.$$
$$\left. h[3] \text{LS}\left[-⟨1⟩ + 2 ⟨2⟩, \frac{13 ⟨12⟩}{2}, \frac{39 ⟨112⟩}{4} - \frac{⟨122⟩}{2}\right]\right]$$

True

Testing hta

```
n = $SeriesShowDegree = $SeriesCompareDegree = 4;
Print /@ {λ0 = Randomλ[3, 2, 2], λ0 // hta[1, 1]};
```

λ[CWS[2 CW[1], -CW[12], -2 CW[122], 0],
 h[1] LS[⟨2⟩, ⟨12⟩, ⟨112⟩ - 2 ⟨122⟩, 0] + h[2] LS[-2 ⟨1⟩ - 2 ⟨2⟩, ⟨12⟩, -⟨122⟩, 0]]

$$\lambda\left[0, h[1] \text{LS}\left[⟨2⟩, ⟨12⟩, ⟨112⟩ - 3 ⟨122⟩, -2 ⟨1122⟩ + \frac{5 ⟨1222⟩}{2}\right] +\right.$$
$$\left. h[2] \text{LS}\left[-2 ⟨1⟩ - 2 ⟨2⟩, 3 ⟨12⟩, 2 ⟨112⟩ - 3 ⟨122⟩, 2 ⟨1112⟩ - 8 ⟨1122⟩ + \frac{11 ⟨1222⟩}{6}\right]\right]$$

```
Print /@ {λ0 = Randomλ[n, 3, 2],
   t1 = λ0 // hta[1, 1] // hta[1, 2] // tm[1, 2, 1],
   t2 = λ0 // tm[1, 2, 1] // hta[1, 1],
   t1 ≡ t2
  };
```

λ[CWS[-2 CW[1] - 2 CW[2] - 2 CW[3], CW[13] + 2 CW[23],
  2 CW[112] + 2 CW[113] + 2 CW[122] + 2 CW[132] + 2 CW[133] - CW[233],
  CW[1112] + CW[1113] + CW[1122] + CW[1123] + 2 CW[1132] - CW[1133] - CW[1222] - 2 CW[1223] -
   CW[1232] + CW[1322] - CW[1323] + 2 CW[1332] - 2 CW[1333] + 2 CW[2233] + 2 CW[2333]],
 h[1] LS[-2 ⟨1⟩ + 2 ⟨2⟩ - 2 ⟨3⟩, -2 ⟨12⟩ - 2 ⟨13⟩,
    2 ⟨112⟩ - 2 ⟨113⟩ + 2 ⟨122⟩ - 2 ⟨123⟩ + 2 ⟨132⟩ + ⟨133⟩ - ⟨223⟩ + ⟨233⟩,
    -2 ⟨1112⟩ + 2 ⟨1113⟩ - ⟨1123⟩ + ⟨1133⟩ + 2 ⟨1213⟩ - 2 ⟨1222⟩ + ⟨1223⟩ - 2 ⟨1232⟩ -
     ⟨1233⟩ - ⟨1322⟩ - 2 ⟨1323⟩ + 2 ⟨1332⟩ - 2 ⟨1333⟩ + 2 ⟨2223⟩ - ⟨2233⟩ + ⟨2333⟩] +
  h[2] LS[⟨2⟩ + ⟨3⟩, -2 ⟨12⟩ - 2 ⟨13⟩ - ⟨23⟩, 2 ⟨112⟩ - 2 ⟨113⟩ + 2 ⟨123⟩ - ⟨133⟩ +
     2 ⟨223⟩ + ⟨233⟩, ⟨1112⟩ - ⟨1132⟩ + ⟨1133⟩ - 2 ⟨1213⟩ + 2 ⟨1222⟩ - ⟨1223⟩ -
     2 ⟨1232⟩ + 2 ⟨1233⟩ - 2 ⟨1322⟩ + 2 ⟨1323⟩ + ⟨1332⟩ - 2 ⟨1333⟩ - 2 ⟨2223⟩ + ⟨2233⟩]]

$$\lambda\left[0, h[1] \text{LS}[-2 ⟨3⟩, -2 ⟨13⟩, -7 ⟨113⟩ - 2 ⟨133⟩, 5 ⟨1113⟩ - 21 ⟨1133⟩ - ⟨1333⟩] +\right.$$
$$\left. h[2] \text{LS}\left[⟨1⟩ + ⟨3⟩, -⟨13⟩, 4 ⟨113⟩ - 4 ⟨133⟩, 5 ⟨1113⟩ + 5 ⟨1133⟩ - \frac{20 ⟨1333⟩}{3}\right]\right]$$

$$\lambda\left[0, h[1] \text{LS}[-2 ⟨3⟩, -2 ⟨13⟩, -7 ⟨113⟩ - 2 ⟨133⟩, 5 ⟨1113⟩ - 21 ⟨1133⟩ - ⟨1333⟩] +\right.$$
$$\left. h[2] \text{LS}\left[⟨1⟩ + ⟨3⟩, -⟨13⟩, 4 ⟨113⟩ - 4 ⟨133⟩, 5 ⟨1113⟩ + 5 ⟨1133⟩ - \frac{20 ⟨1333⟩}{3}\right]\right]$$

0 ≡ 0

```
n = $SeriesShowDegree = $SeriesCompareDegree = 4;
Print /@ {λ0 = Randomλ[n, 2, 3],
    t1 = λ0 // hta[1, 1] // hta[2, 1] // hm[1, 2, 1],
    t2 = λ0 // hm[1, 2, 1] // hta[1, 1],
    t1 ≡ t2
   };
```

$\lambda[\text{CWS}[-\text{CW}[1] - \text{CW}[2], -2\,\text{CW}[12], 2\,\text{CW}[112] - \text{CW}[122], \text{CW}[1112] + 2\,\text{CW}[1122] - \text{CW}[1222]],$
$\quad h[1]\,\text{LS}[2\,\langle 2\rangle, -\langle 12\rangle, -\langle 112\rangle + 2\,\langle 122\rangle, \langle 1122\rangle - 2\,\langle 1222\rangle] +$
$\quad\quad h[2]\,\text{LS}[-2\,\langle 1\rangle - \langle 2\rangle, 2\,\langle 12\rangle, -\langle 112\rangle - \langle 122\rangle, \langle 1122\rangle + 2\,\langle 1222\rangle] +$
$\quad\quad h[3]\,\text{LS}[-2\,\langle 1\rangle - \langle 2\rangle, -\langle 12\rangle, -\langle 112\rangle - \langle 122\rangle, -\langle 1112\rangle - \langle 1122\rangle - \langle 1222\rangle]]$

$\lambda\Big[0, h[3]\,\text{LS}\Big[-2\,\langle 1\rangle - \langle 2\rangle, \langle 12\rangle, 7\,\langle 112\rangle - \langle 122\rangle, \dfrac{5\,\langle 1112\rangle}{3} - \dfrac{32\,\langle 1122\rangle}{3} - \dfrac{\langle 1222\rangle}{6}\Big] +$
$\quad h[1]\,\text{LS}\Big[-2\,\langle 1\rangle + \langle 2\rangle, 5\,\langle 12\rangle, \dfrac{17\,\langle 112\rangle}{3} - \dfrac{11\,\langle 122\rangle}{2}, \dfrac{4\,\langle 1112\rangle}{3} - \dfrac{39\,\langle 1122\rangle}{2} + \dfrac{53\,\langle 1222\rangle}{12}\Big]\Big]$

$\lambda\Big[0, h[1]\,\text{LS}\Big[-2\,\langle 1\rangle + \langle 2\rangle, 5\,\langle 12\rangle, \dfrac{17\,\langle 112\rangle}{3} - \dfrac{11\,\langle 122\rangle}{2}, \dfrac{4\,\langle 1112\rangle}{3} - \dfrac{39\,\langle 1122\rangle}{2} + \dfrac{53\,\langle 1222\rangle}{12}\Big] +$
$\quad h[3]\,\text{LS}\Big[-2\,\langle 1\rangle - \langle 2\rangle, \langle 12\rangle, 7\,\langle 112\rangle - \langle 122\rangle, \dfrac{5\,\langle 1112\rangle}{3} - \dfrac{32\,\langle 1122\rangle}{3} - \dfrac{\langle 1222\rangle}{6}\Big]\Big]$

$0 \equiv 0$

Testing dm

```
$SeriesShowDegree = 3;
$SeriesCompareDegree = n = 6;
Timing[Print /@ {λ0 = Randomλ[n, 4, 4],
    t1 = λ0 // dm[1, 2, 1] // dm[1, 3, 1],
    t2 = λ0 // dm[2, 3, 2] // dm[1, 2, 1],
    t1 ≡ t2
   };]
```

λ[CWS[CW[1] + CW[2] − 2 CW[3] + CW[4], −2 CW[12] − 2 CW[14] − CW[23] + 2 CW[24] + 2 CW[34],
    2 CW[112] − 2 CW[113] − 2 CW[122] + 2 CW[132] − 2 CW[134] + 2 CW[142] −
      2 CW[143] − 2 CW[144] − CW[224] + CW[234] − CW[243] + CW[244] + 2 CW[344]],
  h[1] LS[−2 ⟨1⟩ − 2 ⟨2⟩ + ⟨4⟩, −2 ⟨12⟩ + ⟨13⟩ − 2 ⟨14⟩ − ⟨23⟩ − 2 ⟨34⟩,
      −2 ⟨113⟩ + 2 ⟨114⟩ − 2 ⟨123⟩ + 2 ⟨124⟩ − ⟨132⟩ + 2 ⟨133⟩ + ⟨134⟩ + ⟨142⟩ −
        ⟨143⟩ + ⟨144⟩ + 2 ⟨223⟩ − ⟨224⟩ + ⟨243⟩ + ⟨244⟩ + ⟨334⟩ + ⟨344⟩] +
    h[2] LS[−2 ⟨1⟩ + 2 ⟨2⟩ − 2 ⟨3⟩ + 2 ⟨4⟩, ⟨12⟩ − 2 ⟨13⟩ + 2 ⟨14⟩ + 2 ⟨23⟩ − ⟨24⟩ − ⟨34⟩,
        ⟨122⟩ − 2 ⟨123⟩ − 2 ⟨124⟩ + ⟨132⟩ − 2 ⟨133⟩ + 2 ⟨142⟩ + ⟨144⟩ + ⟨223⟩ − ⟨224⟩ + 2 ⟨233⟩ −
          2 ⟨243⟩ + 2 ⟨244⟩ + ⟨334⟩ + ⟨344⟩] + h[3] LS[−2 ⟨1⟩ − 2 ⟨4⟩, ⟨12⟩ + ⟨13⟩ + ⟨14⟩ − 2 ⟨24⟩,
        −⟨112⟩ + 2 ⟨114⟩ + 2 ⟨122⟩ − 2 ⟨123⟩ + 2 ⟨124⟩ + 2 ⟨133⟩ − 2 ⟨134⟩ − 2 ⟨142⟩ + ⟨143⟩ +
          ⟨144⟩ + 2 ⟨223⟩ − 2 ⟨224⟩ + 2 ⟨233⟩ + 2 ⟨234⟩ − ⟨243⟩ − ⟨244⟩ − 2 ⟨334⟩ − ⟨344⟩] +
    h[4] LS[2 ⟨1⟩ − ⟨2⟩ + 2 ⟨3⟩ − ⟨4⟩, 2 ⟨12⟩ + ⟨13⟩ + 2 ⟨14⟩ + ⟨23⟩ − 2 ⟨24⟩ + ⟨34⟩,
        2 ⟨112⟩ + ⟨114⟩ + ⟨122⟩ + ⟨123⟩ + 2 ⟨124⟩ + 2 ⟨132⟩ + ⟨133⟩ + ⟨142⟩ + ⟨143⟩ +
          2 ⟨223⟩ + 2 ⟨224⟩ + ⟨233⟩ − 2 ⟨234⟩ + 2 ⟨243⟩ + 2 ⟨244⟩ + 2 ⟨334⟩ + ⟨344⟩]]

$\lambda\Big[0$, h[4] LS[3 ⟨1⟩ − ⟨4⟩, −⟨14⟩, −10 ⟨114⟩ − 3 ⟨144⟩] +

  h[1] LS$\Big[−8$ ⟨1⟩ + ⟨4⟩, ⟨14⟩, 19 ⟨114⟩ − $\dfrac{19 ⟨144⟩}{2}\Big]\Big]$

$\lambda\Big[0$, h[4] LS[3 ⟨1⟩ − ⟨4⟩, −⟨14⟩, −10 ⟨114⟩ − 3 ⟨144⟩] +

  h[1] LS$\Big[−8$ ⟨1⟩ + ⟨4⟩, ⟨14⟩, 19 ⟨114⟩ − $\dfrac{19 ⟨144⟩}{2}\Big]\Big]$

$0 \equiv 0$

{4.244, Null}

λ1 = λ0 /. h[1] s_LieSeries :→ h[1] ScaleLieSeries[t, s]

λ[CWS[CW[1] − 2 CW[2] + CW[3] − CW[4], −CW[13] + 2 CW[14] + 2 CW[23] + 2 CW[24] + 2 CW[34],
    2 CW[112] − 2 CW[114] − 2 CW[122] − CW[123] + CW[124] − CW[132] −
      CW[133] − 2 CW[134] + 2 CW[142] + 2 CW[143] + 2 CW[144] − 2 CW[223] −
      2 CW[224] − CW[233] − 2 CW[234] + CW[244] − CW[334] + 2 CW[344]],
  h[2] LS[⟨1⟩ − ⟨2⟩ − ⟨3⟩ − 2 ⟨4⟩, 2 ⟨12⟩ + 2 ⟨13⟩ + 2 ⟨14⟩ + ⟨23⟩ + ⟨24⟩ − 2 ⟨34⟩,
      −⟨113⟩ − ⟨114⟩ − ⟨123⟩ + 2 ⟨124⟩ − ⟨133⟩ − ⟨134⟩ + 2 ⟨142⟩ − 2 ⟨143⟩ + ⟨223⟩ +
        ⟨224⟩ − 2 ⟨234⟩ + 2 ⟨243⟩ + ⟨334⟩ − 2 ⟨344⟩] + h[3] LS[2 ⟨1⟩ − 2 ⟨2⟩ + ⟨4⟩,
      2 ⟨12⟩ − ⟨13⟩ − 2 ⟨23⟩ + ⟨24⟩, −⟨112⟩ − ⟨113⟩ − 2 ⟨114⟩ + ⟨122⟩ − 2 ⟨123⟩ − 2 ⟨124⟩ +
        2 ⟨132⟩ + ⟨142⟩ + 2 ⟨223⟩ − 2 ⟨224⟩ − 2 ⟨233⟩ + 2 ⟨234⟩ − 2 ⟨244⟩ + 2 ⟨334⟩ + ⟨344⟩] +
    h[4] LS[⟨1⟩ + 2 ⟨2⟩ + 2 ⟨3⟩, ⟨12⟩ + ⟨23⟩ + ⟨34⟩, −⟨114⟩ − ⟨122⟩ − 2 ⟨123⟩ + ⟨132⟩ −
        ⟨134⟩ − ⟨142⟩ − 2 ⟨143⟩ + ⟨144⟩ + ⟨223⟩ + 2 ⟨224⟩ + 2 ⟨233⟩ − 2 ⟨234⟩ + ⟨243⟩ − ⟨334⟩] +
    h[1] LS[2 t ⟨1⟩ + t ⟨2⟩ − t ⟨3⟩, −2 t ⟨13⟩ + t ⟨14⟩ − ⟨24⟩ + 2 t ⟨34⟩,
      −2 t ⟨112⟩ + 2 t ⟨113⟩ − t ⟨122⟩ − 2 t ⟨123⟩ + t ⟨124⟩ − 2 t ⟨132⟩ + 2 t ⟨133⟩ + 2 t ⟨134⟩ −
        t ⟨142⟩ − 2 t ⟨144⟩ + 2 t ⟨223⟩ − t ⟨224⟩ − t ⟨233⟩ − 2 t ⟨234⟩ + t ⟨243⟩ + t ⟨244⟩ + 2 t ⟨344⟩]]

```
t1 = λ1 // hta[1, 1]
```

$\lambda[0, h[2] \, LS[\langle 1 \rangle - \langle 2 \rangle - \langle 3 \rangle - 2\langle 4 \rangle,$
$3\langle 1 \rangle + 2\langle 12 \rangle + 2\langle 13 \rangle + 2\langle 14 \rangle - 3\langle 2 \rangle + \langle 23 \rangle + \langle 24 \rangle - 3\langle 3 \rangle - 2\langle 34 \rangle - 6\langle 4 \rangle,$
$9\langle 1 \rangle - \langle 113 \rangle - \langle 114 \rangle + 8\langle 12 \rangle - \langle 123 \rangle + 2\langle 124 \rangle + 8\langle 13 \rangle - \langle 133 \rangle - \langle 134 \rangle + 8\langle 14 \rangle +$
$2\langle 142 \rangle - 2\langle 143 \rangle - 9\langle 2 \rangle + \langle 223 \rangle + \langle 224 \rangle + 4\langle 23 \rangle - 2\langle 234 \rangle +$
$4\langle 24 \rangle + 2\langle 243 \rangle - 9\langle 3 \rangle + \langle 334 \rangle - 8\langle 34 \rangle - 2\langle 344 \rangle - 18\langle 4 \rangle] +$
$h[3] \, LS[2\langle 1 \rangle - 2\langle 2 \rangle + \langle 4 \rangle, 6\langle 1 \rangle + 2\langle 12 \rangle - \langle 13 \rangle - 6\langle 2 \rangle - 2\langle 23 \rangle + \langle 24 \rangle + 3\langle 4 \rangle, 18\langle 1 \rangle - \langle 112 \rangle -$
$\langle 113 \rangle - 2\langle 114 \rangle + 8\langle 12 \rangle + \langle 122 \rangle - 2\langle 123 \rangle - 2\langle 124 \rangle - 4\langle 13 \rangle + 2\langle 132 \rangle + \langle 142 \rangle - 18\langle 2 \rangle +$
$2\langle 223 \rangle - 2\langle 224 \rangle - 8\langle 23 \rangle - 2\langle 233 \rangle + 2\langle 234 \rangle + 4\langle 24 \rangle - 2\langle 244 \rangle + 2\langle 334 \rangle + \langle 344 \rangle + 9\langle 4 \rangle] +$
$h[4] \, LS[\langle 1 \rangle + 2\langle 2 \rangle + 2\langle 3 \rangle, 3\langle 1 \rangle + \langle 12 \rangle + 6\langle 2 \rangle + \langle 23 \rangle + 6\langle 3 \rangle + \langle 34 \rangle,$
$9\langle 1 \rangle - \langle 114 \rangle + 4\langle 12 \rangle - \langle 122 \rangle - 2\langle 123 \rangle + \langle 132 \rangle - \langle 134 \rangle - \langle 142 \rangle - 2\langle 143 \rangle + \langle 144 \rangle +$
$18\langle 2 \rangle + \langle 223 \rangle + 2\langle 224 \rangle + 4\langle 23 \rangle + 2\langle 233 \rangle - 2\langle 234 \rangle + \langle 243 \rangle + 18\langle 3 \rangle - \langle 334 \rangle + 4\langle 34 \rangle] +$
$h[1] \, LS[2t\langle 1 \rangle + t\langle 2 \rangle - t\langle 3 \rangle, 6t\langle 1 \rangle - 2t\langle 13 \rangle + t\langle 14 \rangle + 3t\langle 2 \rangle - t\langle 24 \rangle - 3t\langle 3 \rangle + 2t\langle 34 \rangle,$
$18t\langle 1 \rangle - 2t\langle 112 \rangle + 2t\langle 113 \rangle - t\langle 122 \rangle - 2t\langle 123 \rangle + t\langle 124 \rangle - 8t\langle 13 \rangle - 2t\langle 132 \rangle +$
$2t\langle 133 \rangle + 2t\langle 134 \rangle + 4t\langle 14 \rangle - t\langle 142 \rangle - 2t\langle 144 \rangle + 9t\langle 2 \rangle + 2t\langle 223 \rangle - t\langle 224 \rangle -$
$t\langle 233 \rangle - 2t\langle 234 \rangle - 4t\langle 24 \rangle + t\langle 243 \rangle + t\langle 244 \rangle - 9t\langle 3 \rangle + 8t\langle 34 \rangle + 2t\langle 344 \rangle]]$

```
Clear[t2, t3];
t2[d_] := t1 /. s_LieSeries :→ D[s[d], t];
μx = D[t1[[2]], h[1]];
der = LieDerivation[{LW[1] → b[ScaleLieSeries[1 / t, μx], LW[1]]}];
t3 = λ[
   t1[[1]] // der,
   (t1[[2]] /. {h[1] → 0, s_LieSeries :→ der[s]}) +
    h[1] (ScaleLieSeries[1 / t, μx] + (μx // der))
   ];
{μx, t2[1], t3[1]}
```

$\{LS[2t\langle 1 \rangle + t\langle 2 \rangle - t\langle 3 \rangle, 6t\langle 1 \rangle - 2t\langle 13 \rangle + t\langle 14 \rangle + 3t\langle 2 \rangle - t\langle 24 \rangle - 3t\langle 3 \rangle + 2t\langle 34 \rangle,$
$18t\langle 1 \rangle - 2t\langle 112 \rangle + 2t\langle 113 \rangle - t\langle 122 \rangle - 2t\langle 123 \rangle + t\langle 124 \rangle - 8t\langle 13 \rangle - 2t\langle 132 \rangle +$
$2t\langle 133 \rangle + 2t\langle 134 \rangle + 4t\langle 14 \rangle - t\langle 142 \rangle - 2t\langle 144 \rangle + 9t\langle 2 \rangle + 2t\langle 223 \rangle - t\langle 224 \rangle -$
$t\langle 233 \rangle - 2t\langle 234 \rangle - 4t\langle 24 \rangle + t\langle 243 \rangle + t\langle 244 \rangle - 9t\langle 3 \rangle + 8t\langle 34 \rangle + 2t\langle 344 \rangle],$
$\lambda[0, h[1] (2\langle 1 \rangle + \langle 2 \rangle - \langle 3 \rangle)], \lambda[0, h[1] (2\langle 1 \rangle + \langle 2 \rangle - \langle 3 \rangle)]\}$

```
n = 2; Print /@ {t2[n], t3[n], t2[n] == t3[n]};
```

$\lambda[0, h[1] (6\langle 1 \rangle - 2\langle 13 \rangle + \langle 14 \rangle + 3\langle 2 \rangle - \langle 24 \rangle - 3\langle 3 \rangle + 2\langle 34 \rangle)]$

$\lambda[0, h[2] (-4\langle 12 \rangle + 4\langle 13 \rangle) + h[4] (-4\langle 12 \rangle + 4\langle 13 \rangle) + h[3] (-8\langle 12 \rangle + 8\langle 13 \rangle) +$
$h[1] (6\langle 1 \rangle - 8t\langle 12 \rangle - 2\langle 13 \rangle + 8t\langle 13 \rangle + \langle 14 \rangle + 3\langle 2 \rangle - \langle 24 \rangle - 3\langle 3 \rangle + 2\langle 34 \rangle)]$

$\lambda[0, h[1] (6\langle 1 \rangle - 2\langle 13 \rangle + \langle 14 \rangle + 3\langle 2 \rangle - \langle 24 \rangle - 3\langle 3 \rangle + 2\langle 34 \rangle)] ==$
$\lambda[0, h[2] (-4\langle 12 \rangle + 4\langle 13 \rangle) + h[4] (-4\langle 12 \rangle + 4\langle 13 \rangle) + h[3] (-8\langle 12 \rangle + 8\langle 13 \rangle) +$
$h[1] (6\langle 1 \rangle - 8t\langle 12 \rangle - 2\langle 13 \rangle + 8t\langle 13 \rangle + \langle 14 \rangle + 3\langle 2 \rangle - \langle 24 \rangle - 3\langle 3 \rangle + 2\langle 34 \rangle)]$

```
n = 3; t2[n] == t3[n]
```

$\lambda[0, h[1] \, (18 \, \langle 1 \rangle - 2 \, \langle 112 \rangle + 2 \, \langle 113 \rangle - \langle 122 \rangle - 2 \, \langle 123 \rangle + \langle 124 \rangle -$
$\qquad 8 \, \langle 13 \rangle - 2 \, \langle 132 \rangle + 2 \, \langle 133 \rangle + 2 \, \langle 134 \rangle + 4 \, \langle 14 \rangle - \langle 142 \rangle - 2 \, \langle 144 \rangle + 9 \, \langle 2 \rangle + 2 \, \langle 223 \rangle -$
$\qquad \langle 224 \rangle - \langle 233 \rangle - 2 \, \langle 234 \rangle - 4 \, \langle 24 \rangle + \langle 243 \rangle + \langle 244 \rangle - 9 \, \langle 3 \rangle + 8 \, \langle 34 \rangle + 2 \, \langle 344 \rangle)] \, ==$
$\quad \lambda[0, h[3] \, (52 \, \langle 113 \rangle - 26 \, \langle 114 \rangle - 78 \, \langle 12 \rangle - 10 \, \langle 122 \rangle + 5 \, \langle 123 \rangle +$
$\qquad 26 \, \langle 124 \rangle + 78 \, \langle 13 \rangle + 15 \, \langle 132 \rangle - 5 \, \langle 133 \rangle - 52 \, \langle 134 \rangle) +$
$\qquad h[4] \, (26 \, \langle 113 \rangle - 13 \, \langle 114 \rangle - 39 \, \langle 12 \rangle - 5 \, \langle 122 \rangle + 13 \, \langle 124 \rangle + 39 \, \langle 13 \rangle + 5 \, \langle 132 \rangle - 26 \, \langle 134 \rangle) +$
$\qquad h[2] \, (26 \, \langle 113 \rangle - 13 \, \langle 114 \rangle - 39 \, \langle 12 \rangle - 10 \, \langle 122 \rangle - 10 \, \langle 123 \rangle +$
$\qquad 3 \, \langle 124 \rangle + 39 \, \langle 13 \rangle + 10 \, \langle 133 \rangle - 16 \, \langle 134 \rangle - 10 \, \langle 142 \rangle + 10 \, \langle 143 \rangle) +$
$\qquad h[1] \, (18 \, \langle 1 \rangle - 2 \, \langle 112 \rangle + 2 \, \langle 113 \rangle + 52 \, t \, \langle 113 \rangle - 26 \, t \, \langle 114 \rangle - 78 \, t \, \langle 12 \rangle - \langle 122 \rangle - 2 \, \langle 123 \rangle +$
$\qquad 10 \, t \, \langle 123 \rangle + \langle 124 \rangle + 21 \, t \, \langle 124 \rangle - 8 \, \langle 13 \rangle + 78 \, t \, \langle 13 \rangle - 2 \, \langle 132 \rangle + 10 \, t \, \langle 132 \rangle + 2 \, \langle 133 \rangle -$
$\qquad 10 \, t \, \langle 133 \rangle + 2 \, \langle 134 \rangle - 47 \, t \, \langle 134 \rangle + 4 \, \langle 14 \rangle - \langle 142 \rangle - 5 \, t \, \langle 142 \rangle + 5 \, t \, \langle 143 \rangle - 2 \, \langle 144 \rangle + 9 \, \langle 2 \rangle +$
$\qquad 2 \, \langle 223 \rangle - \langle 224 \rangle - \langle 233 \rangle - 2 \, \langle 234 \rangle - 4 \, \langle 24 \rangle + \langle 243 \rangle + \langle 244 \rangle - 9 \, \langle 3 \rangle + 8 \, \langle 34 \rangle + 2 \, \langle 344 \rangle)]$

```
Table[t2[n] == t3[n], {n, 6}]
```

A very large output was generated. Here is a sample of it:

```
{True, ≪4≫,
 λ[0, h[1] (702 ⟨1⟩ - 2 ⟨111112⟩ - 2 ⟨111114⟩ - 7 ⟨11112⟩ + 2 ⟨111122⟩ + ⟨111123⟩ - ⟨111124⟩ +
      7 ⟨11113⟩ + 2 ⟨111133⟩ + ⟨111134⟩ - 7 ⟨11114⟩ - ⟨111142⟩ - ⟨111143⟩ + ≪1147≫ +
      ⟨333444⟩ + 14 ⟨33434⟩ + ⟨334344⟩ + 54 ⟨3344⟩ - 2 ⟨334434⟩ - 14 ⟨33444⟩ + 2 ⟨334444⟩ +
      350 ⟨34⟩ + 14 ⟨34344⟩ + 152 ⟨344⟩ - 27 ⟨3444⟩ - 2 ⟨344444⟩)] == λ[0, ≪1≫]}
```

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